A note on estimating the false discovery rate

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Summary

Storey's (2002) fixed rejection region approach to multiple testing exercising a control over the false discovery rate (FDR) is based on a conservatively biased point estimate of the FDR. He proposed this estimate assuming a mixture model and considering a fixed rejection threshold for each hypothesis. In this note, we propose an alternative point estimate of the FDR, with a proof that it has uniformly smaller conservative bias than the corresponding Storey's estimate and a numerical evidence that the mean squared error (MSE) is also often smaller. The present estimate provides a more powerful fixed rejection region approach to multiple testing than Storey's estimate and makes a connection, as Storey's estimate does, between this approach and that of Benjamini and Hochberg (1995).

Some key words: Estimation of false discovery rate; UMVU estimate; Multiple testing.

1. Introduction

The false discovery rate (FDR) introduced by Benjamini and Hochberg (1995) has now become a standard error controlling measure in a wide variety of applications involving multiple testing. With V_n and R_n representing the total numbers of rejections and false rejections of null hypotheses respectively while testing n null hypotheses, it is defined as $\mathrm{FDR}_n =$

$E[(V_n/R_n)I(R_n > 0)]$, where I is the indicator function. There are basically two approaches to controlling the FDR. In one, the so called fixed error rate approach originally taken by Benjamini and Hochberg (1995), a rejection region is determined with the error rate, in this case the FDR_n, below a fixed level, as in many traditional multiple testing procedures. In the other, known as the fixed rejection region approach proposed by Storey (2002), the FDR_n is determined through its estimate for a fixed rejection region.

With the observed p-value p_i associated with the ith null hypothesis H_i, Storey (2002) put forward his approach centered on a class of point estimates of the FDR_n for a single-step test rejecting each H_i if $p_i \le \gamma$, for some some fixed threshold γ, proposed under the following model:

Mixture Model. *Let P_i denote the random p-value corresponding to p_i and $H_i = 0$ or 1 according as the associated null hypothesis is true or false. Let (P_i, H_i), $i = 1, \ldots, n$, be independently and identically distributed with* $\Pr(P_i \le u \mid H_i) = (1 - H_i)u + H_i F_1(u)$, $u \in (0,1)$, *for some continuous cdf $F_1(u)$, and* $\Pr(H_i = 0) = \pi_0 = 1 - \Pr(H_i = 1)$.

Storey (2002) considered the following class of point estimates of the FDR_n

$$\widehat{\text{FDR}}_{n,\lambda}(\gamma) = \frac{n\hat{\pi}_0(\lambda)\gamma}{\max\{R_n(\gamma), 1\}}, \ \lambda \in [0,1), \tag{1}$$

where

$$R_n(\gamma) = \sum_{i=1}^{n} I(P_i \le \gamma), \quad \hat{\pi}_0(\lambda) = \frac{n - R_n(\lambda)}{n(1-\lambda)}. \tag{2}$$

In this note, we propose an alternative class of point estimates of the FDR_n with better finite sample properties. We prove that each estimate in the proposed class has uniformly smaller conservative bias than the corresponding estimate in (1), with numerical evidence that the MSE is also quite often smaller. The fixed rejection region approach to multiple testing based on an estimate in the present class is more powerful than using the corresponding estimate in (1). Moreover, a connection between the two approaches to multiple testing is offered through our estimate, as Storey (2002) did through his.

2. The New estimates of the FDR_n

In this section, we present our proposed new point estimates of the FDR_n. These are derived from an identity for the FDR_n given in the following theorem.

Theorem 1. *Under the mixture model, the FDR_n of a single-step test rejecting each H_i if $p_i \le \gamma$, for some fixed $\gamma \in [0,1]$, is given by*

$$\mathrm{FDR}_n(\gamma) = (n+1)\pi_0\gamma \mathrm{E}\left(\frac{1}{\max\{R_n(\gamma), \frac{1}{n}\}+1}\right). \tag{3}$$

Proof. Let $V_n(\gamma) = \sum_{i=1}^n I(P_i \le \gamma, H_i = 0)$, the total number of false rejections. Then, we have

$$\begin{aligned}
\mathrm{FDR}_n(\gamma) &= E\left\{\frac{V_n(\gamma)}{R_n(\gamma)} I\left(R_n(\gamma) > 0\right)\right\} \\
&= E\left\{\sum_{i=1}^n \sum_{r=1}^n \frac{1}{r} I\left(P_i \le \gamma, H_i = 0, R_n(\gamma) = r\right)\right\} \\
&= \sum_{i=1}^n \Pr(P_i \le \gamma, H_i = 0) \sum_{r=1}^n \frac{1}{r} \Pr(R_{n-1}^{(-i)}(\gamma) = r-1) \\
&= \pi_0\gamma\mathrm{E}\left(\sum_{i=1}^n \frac{1}{R_{n-1}^{(-i)}(\gamma)+1}\right),
\end{aligned} \tag{4}$$

where $R_{n-1}^{(-i)}(\gamma) = \sum_{j(\ne i)=1}^n I(P_j \le \gamma)$. The summation inside the expectation can be expressed in terms of $R_n(\gamma)$. When $R_n(\gamma) > 0$, since $R_{n-1}^{(-i)}(\gamma) = R_n(\gamma) - I(P_i \le \gamma)$, it is equal to

$$\begin{aligned}
\sum_{i=1}^n \frac{1}{R_{n-1}^{(-i)}(\gamma)+1} &= \sum_{i:P_i\le\gamma} \frac{1}{R_{n-1}^{(-i)}(\gamma)+1} + \sum_{i:P_i>\gamma} \frac{1}{R_{n-1}^{(-i)}(\gamma)+1} \\
&= \frac{R_n(\gamma)}{R_n(\gamma)} + \frac{n - R_n(\gamma)}{R_n(\gamma)+1} = \frac{n+1}{R_n(\gamma)+1}.
\end{aligned} \tag{5}$$

When $R_n(\gamma) = 0$, since $R_{n-1}^{(-i)}(\gamma) = 0$, it is equal to n. In other words, the right-hand side in (4) is equal to

$$(n+1)\pi_0\gamma E\left(\frac{1}{\max\{R_n(\gamma), \frac{1}{n}\}+1}\right), \tag{6}$$

the right-hand side in (3), which proves the theorem.

It is easy to see that the expression (3) for $\mathrm{FDR}_n(\gamma)$ is same as the alternative expression

$$\mathrm{FDR}_n(\gamma) = \frac{\pi_0 t}{F(t)} \Pr\left(R_n(\gamma) > 0\right), \tag{7}$$

given in Storey (2002) and commonly seen in the literature. Nevertheless, unlike Storey's, the present expression provides directly an idea how to unbiasedly estimate the $\mathrm{FDR}_n(\gamma)$ with a known π_0. This estimate is given by

$$\widehat{\mathrm{FDR}}_n(\gamma|\pi_0) = \frac{(n+1)\pi_0\gamma}{\max\{R_n(\gamma), \frac{1}{n}\} + 1} \ . \tag{8}$$

In fact, since $R_n(\gamma)$ is complete sufficient for $F(\gamma)$, the parameter in the distributions of $(P_1, \ldots, P_n)$, when π_0 is known, (8) is the uniformly minimum variance unbiased estimate (UMVUE) of the FDR_n with a known π_0 based on $(P_1, \ldots, P_n)$.

With unknown π_0, we estimate it as in Storey (2002) using (2), and propose the following class of estimates of the $\mathrm{FDR}_n(\gamma)$:

$$\begin{aligned} \widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma) &= \frac{(n+1)\hat{\pi}_0(\lambda)\gamma}{\max\{R_n(\gamma), \frac{1}{n}\} + 1} \\ &= \frac{(n+1)\left[n - R_n(\lambda)\right]\gamma}{n(1-\lambda)\left[\max\{R_n(\gamma), \frac{1}{n}\} + 1\right]}, \ \lambda \in (0.1]. \end{aligned} \tag{9}$$

The proposed estimate $\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)$ has better finite sample properties than the estimate $\widehat{\mathrm{FDR}}_{n,\lambda}(\gamma)$ given in Storey (2002) for any fixed λ and γ, though they become asymptotically equivalent as $n \to \infty$. More specifically, we have the following theorem.

Theorem 2. *For any fixed* $\lambda, \gamma \in [0, 1]$,

$$\mathrm{FDR}_n(\gamma) \le \mathrm{E}\left(\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)\right) \le \mathrm{E}\left(\widehat{\mathrm{FDR}}_{n,\lambda}(\gamma)\right). \tag{10}$$

Proof. Let us suppress the subscript n in R_n.

$$\mathrm{E}\left(\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)\right) = (n+1)\gamma \mathrm{E}\left\{\hat{\pi}_0(\lambda)\mathrm{E}\left(\frac{1}{\max\{R(\gamma), \frac{1}{n}\}+1}\middle| R(\lambda)\right)\right\}. \tag{11}$$

Since, given $R(\lambda)$,

$$R(\gamma) \sim \text{Binomial}\left(\frac{F(\gamma)}{F(\lambda)},\ R(\lambda)\right),\ \text{when } \gamma \leq \lambda,$$

and

$$n - R(\gamma) \sim \text{Binomial}\left(\frac{1-F(\gamma)}{1-F(\lambda)},\ n - R(\lambda)\right),\ \text{when } \gamma \geq \lambda,$$

$R(\gamma)$ is stochastically increasing in $R(\lambda)$. Therefore, the conditional expectation given $R(\lambda)$ of any decreasing function of $R(\gamma)$, in particular $1/\left[\max\{R(\gamma), \frac{1}{n}\}+1\right]$, is a decreasing function of $R(\lambda)$. As $\hat{\pi}_0(\lambda)$ is also a decreasing function of $R(\lambda)$, it is positively correlated with this conditional expectation, which implies that the outer expectation in (11) is greater than or equal to

$$\begin{aligned} & \mathrm{E}\{\hat{\pi}_0(\lambda)\}\,\mathrm{E}\left\{\mathrm{E}\left(\frac{1}{\max\{R(\gamma), \frac{1}{n}\}+1}\middle| R(\lambda)\right)\right\} \\ = \ & \mathrm{E}\{\hat{\pi}_0(\lambda)\}\,\mathrm{E}\left(\frac{1}{\max\{R(\gamma), \frac{1}{n}\}+1}\right) \geq \pi_0 \mathrm{E}\left(\frac{1}{\max\{R(\gamma), \frac{1}{n}\}+1}\right), \end{aligned} \tag{12}$$

with the inequality following from the fact that $\hat{\pi}_0(\lambda)$ is upward biased as shown in Storey (2002). Thus, we have

$$\mathrm{E}\left(\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)\right) \geq (n+1)\gamma\pi_0 \mathrm{E}\left(\frac{1}{\max\{R(\gamma), \frac{1}{n}\}+1}\right) = \mathrm{FDR}_n(\gamma), \tag{13}$$

the first inequality. The second inequality follows from the fact that $\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma) \leq \widehat{\mathrm{FDR}}_{n,\lambda}(\gamma)$ with probability one.

We will see numerically in the next section that $\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)$ often outperforms $\widehat{\mathrm{FDR}}_{n,\lambda}(\gamma)$ in terms of the MSE. The gain in power from using our estimate of the FDR_n compared to Storey's for multiple testing is clear, because we are controlling a smaller estimated measure ($\widehat{\mathrm{FDR}}^*_{n,\lambda} \leq \widehat{\mathrm{FDR}}_{n,\lambda}$) and thereby allowing more rejections. In other words, between the two thresholds

$$\gamma_\alpha = \sup\left\{0 \leq \gamma \leq 1 : \widehat{\mathrm{FDR}}_{n,\lambda}(\gamma) \leq \alpha\right\} \tag{14}$$

and

$$\gamma_\alpha^* = \sup\left\{0 \leq \gamma \leq 1 : \widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma) \leq \alpha\right\}, \tag{15}$$

given a level α for the FDR_n, one can easily see that γ_α^* is larger than γ_α and hence results in a more powerful multiple testing procedure. A numerical evidence of the extent of this power improvement can be seen in the next section.

3. Simulation study

We present in this section some numerical results of a simulation study investigating the extent of improvement one gets by using our estimate $\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)$ of the FDR_n over Storey's estimate $\widehat{\mathrm{FDR}}_{n,\lambda}(\gamma)$. The investigation was based on (i) the conservative bias and MSE and (ii) the average power of the multiple testing procedure based on the resulting threshold. The average power is defined as the expected proportion of false nulls that are correctly rejected, which for independent tests and a fixed non-random threshold γ reduces simply to the probability $\Pr(p_i \leq \gamma)$ evaluated for a non-null p_i.

As in Storey (2002), we generated $n = 1000$ independent pairs of observations $(X_i, Z_i),\ i = 1, \ldots, 1000$, where Z_i is the outcome of a Bernoulli experiment with $1 - \pi_0$ as the success probability and $X_i \sim N(\mu_i, 1)$, with $\mu_i = \delta Z_i$ for a fixed $\delta > 0$. For each X_i, we calculated the p-value $p_i = 1 - \Phi(X_i)$, where Φ is the cdf of $N(0,1)$, and performed a test for testing $\mu_i = 0$ against $\mu_i = \delta$ based on the rejection region $\{p_i \leq \gamma\}$. This was repeated over 1000 iterations for each combination of values of π_0, γ and δ selected from the range $0.1, 0.2, \ldots, 0.9$, in increments of 0.1, for π_0, the two values 0.001 and 0.01 for γ, and the two values 1 and 2 for δ. We used $\lambda = 1/2$ in all calculations. The formula (7) was used to compute the true FDR_n for the bias and MSE calculations. For power comparison, we simulated the thresholds γ_α and γ_α^* by estimating them in each iteration and averaging each of them over the 1000 iterations before computing the powers. We used $\alpha = 0.05$ and 0.10.

The numerical findings are summarized in Tables 1 and 2, with the comparison in terms of the conservative bias and MSE being presented in Table 1 and that in terms of the average power in Table 2. The proposed estimate of the FDR_n is seen to have significantly less conservative bias and the MSE compared to Storey's estimate, especially when π_0 is not large. For instance, when $\pi_0 = 0.9$ and $\delta = 2$, our estimate reduces the conservative bias in Storey's estimate by 68%, with the corresponding reduction in the MSE being 11%. A reduction of the conservative bias in Storey's estimate of the FDR_n by ours translates into a higher power of the corresponding multiple testing method, as seen in Table 2. The improvement in power is good when π_0 is large.

4. A connection to the Benjamini-Hochberg method

Storey (2002) made a heuristic connection between the two approaches to multiple testing using his estimate $\widehat{\mathrm{FDR}}_{n,\lambda}$. In particular, he pointed out that the fixed error rate approach to controlling the FDR_n at α due to Banjmaini and Hochberg (1995) (referred to as the BH method) is equivalent to thresholding the p-values at $p_{(\hat{l}_\alpha(0))}$, where

$$\hat{l}_\alpha(\lambda) = \max\left\{1 \le i \le n : \widehat{\mathrm{FDR}}_{n,\lambda}(p_{(i)}) \le \alpha\right\} \tag{16}$$

and $p_{(1)} \le \cdots \le p_{(n)}$ are the ordered observed p-values. This equivalence is seen from the fact that $\widehat{\mathrm{FDR}}_{n,\lambda=0}(p_{(i)}) = np_{(i)}/i$ and the BH method is a stepup test with the critical values $\alpha_i = i\alpha/n$, $i = 1, \ldots, n$, rejecting H_i with $p_i \le p_{(k)}$, where $k = \max\left\{1 \le i \le n : p_{(i)} \le \alpha_i\right\}$, provided the maximum exists, otherwise, accepting all hypotheses. Since

$$\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma) = \frac{(n+1)\max\{R_n(\gamma), 1\}}{n[\max\{R_n(\gamma), \frac{1}{n}\} + 1]}\, \widehat{\mathrm{FDR}}_{n,\lambda}(\gamma), \tag{17}$$

this connection can be established using our estimate by simply rewriting the above threshold as follows:

$$\hat{l}_\alpha(\lambda) = \max\left\{1 \le i \le n : \widehat{\mathrm{FDR}}^*_{n,\lambda}(p_{(i)}) \le \frac{i(n+1)\alpha}{(i+1)n}\right\}. \tag{18}$$

Nevertheless, an alternative, more direct connection can be made in terms of our estimate of the FDR_n at each p_i based on the remaining p-values. When π_0 is unknown, the UMVUE of $\text{FDR}_n(\gamma)$ based on the $n-1$ p-values other than the p_i is $\widehat{\text{FDR}}_n^{(-i)}(\gamma|\pi_0) = n\pi_0\gamma/(R_{n-1}^{(-i)}(\gamma)+1)$ (see (4)). This is different from $\widehat{\text{FDR}}_{n-1,\lambda}^{*(-i)}(\gamma)$, the UMVUE of the true FDR without the ith test, the $\text{FDR}_{n-1}(\gamma)$, but can be expressed in terms of it as follows:

$$\widehat{\text{FDR}}_{n,\lambda}^{*(-i)}(\gamma) = \begin{cases} \widehat{\text{FDR}}_{n-1,\lambda}^{*(-i)}(\gamma) & \text{if} \quad R_{n-1}^{(-i)}(\gamma) > 0 \\ n\pi_0\gamma & \text{if} \quad R_{n-1}^{(-i)}(\gamma) = 0. \end{cases} \tag{19}$$

With unknown π_0, we can consider the estimate

$$\widehat{\text{FDR}}_{n,\lambda}^{*(-i)}(\gamma) = \frac{n\hat{\pi}_0(\lambda)\gamma}{R_{n-1}^{(-i)}(\gamma)+1}. \tag{20}$$

For notational convenience, we now denote the ordered p-values as $p_{i_1} \leq \cdots \leq p_{i_n}$. Then, we have the following theorem.

Theorem 3. *Reject H_i if $p_i \leq p_{i_k}$, where*

$$k = \max\left\{1 \leq j \leq n : \widehat{FDR}_{n,\lambda=0}^{*(-i_j)}(p_{i_j}) \leq \alpha\right\},$$

provided the maximum exists, otherwise, accept all hypotheses. The procedure controls the FDR_n at α when the p-values are either independent or positively dependent in the following sense:

$$E\left\{\phi(p_1,\ldots,p_n) \mid p_i\right\} \quad \uparrow p_i, \tag{21}$$

for any null p_i and (coordinatewise) nondecreasing function ϕ.

The procedure in Theorem 3 is same as the BH procedure, the FDR control of which under the conditions stated in the theorem is known; see, for example, Sarkar (2002).

4. Concluding remarks

Storey's (2002) choice of the estimate of the FDR_n is mostly motivated by his consideration of $n\pi_0\gamma/\max\{R_n(\gamma),1\}$ as an estimate of it when π_0 is known that makes a connection between his fixed rejection region approach to multiple testing and the fixed error rate approach of Benjamini and Hochberg (1995), although he remarked that there are possibly better estimates of the FDR_n through which his method could be improved. A considerable amount of research has taken place since then on improving Storey's estimate. Of course, almost all of it has been devoted to improving the estimate through an improved version of the original estimate of π_0 in (2). We, on the other hand, consider improving it through an improved version of the original estimate of the FDR_n with a known π_0. Interestingly, we can use our proposed improvement of Storey's estimate to describe the BH method, as in the case of Storey's original estimate.

Our main focus in this paper has been on estimating the FDR_n, although similar results can be obtained for the related measure of positive false discovery rate (pFDR) defined as $\mathrm{pFDR}_n(\gamma) = \mathrm{FDR}_n(\gamma)/\Pr\{R_n(\gamma) \geq 1\}$ by Storey (2002). Using our proposed estimate $\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)$ of the FDR_n, we can propose the following estimate of the pFDR_n:

$$\widehat{\mathrm{pFDR}}^*_{n,\lambda}(\gamma) = \frac{\widehat{\mathrm{FDR}}^*_{n,\lambda}(\gamma)}{1-(1-\gamma)^n}, \tag{22}$$

as an alternative to Storey's (2002) original estimate

$$\widehat{\mathrm{pFDR}}_{n,\lambda}(\gamma) = \frac{\widehat{\mathrm{FDR}}_{n,\lambda}(\gamma)}{1-(1-\gamma)^n}. \tag{23}$$

Theorem 2 can be easily extended to the pFDR_n showing that our estimate in (22) has better finite sample performance in terms of the conservative bias [under the standard assumption that $F_1(u) \geq u$ for all $u \in (0,1)$] than Storey's in (23). Furthermore, Storey's q-value method can be refined using (22) to obtain a better method of testing based on the pFDR_n.

Methods based on Storey's original estimates of the FDR_n and pFDR_n have received considerable attention in the field of multiple testing. Now that we have more precise estimates of these quantities, we hope researchers will find it interesting to improve those methods using our estimates, at least when the number of hypotheses is not too large. A number of alternative methods of estimating π_0 improving the estimate in (2) have been suggested in the literature. Further improvements of the proposed FDR_n and pFDR_n estimates are possible using these other estimates of π_0.

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Table 1: Numerical comparison in terms of the bias and MSE

	$\delta = 1$					$\delta = 2$				
		Bias		MSE			Bias		MSE($\times 10^{-5}$)	
π_0	FDR	$\widehat{\text{FDR}}_\lambda(t)$	$\widehat{\text{FDR}}^*_\lambda(t)$	$\widehat{\text{FDR}}_\lambda(t)$	$\widehat{\text{FDR}}^*_\lambda(t)$	FDR	$\widehat{\text{FDR}}_\lambda(t)$	$\widehat{\text{FDR}}^*_\lambda(t)$	$\widehat{\text{FDR}}_\lambda(t)$	$\widehat{\text{FDR}}^*_\lambda(t)$
$\gamma = 0.01$										
0.1	0.012	0.03460	0.03408	0.00124	0.00120	0.003	0.00121	0.00120	0.176	0.174
0.2	0.026	0.03433	0.03358	0.00125	0.00120	0.007	0.00126	0.00124	0.238	0.232
0.3	0.044	0.03422	0.03312	0.00130	0.00122	0.011	0.00130	0.00126	0.344	0.334
0.4	0.067	0.03396	0.03233	0.00137	0.00125	0.018	0.00137	0.00130	0.516	0.495
0.5	0.098	0.03378	0.03128	0.00157	0.00138	0.026	0.00131	0.00119	0.875	0.836
0.6	0.140	0.03497	0.03096	0.00212	0.00177	0.039	0.00158	0.00136	1.914	1.816
0.7	0.202	0.03601	0.02914	0.00343	0.00273	0.059	0.00174	0.00128	4.700	4.439
0.8	0.302	0.03850	0.02540	0.00693	0.00526	0.097	0.00282	0.00169	16.047	14.875
0.9	0.494	0.05441	0.02270	0.02492	0.01745	0.195	0.00628	0.00201	113.208	100.684
$\gamma = 0.001$										
0.1	0.006	0.01907	0.01747	0.00041	0.00034	0.0008	0.00033	0.00032	0.014	0.013
0.2	0.013	0.01978	0.01738	0.00049	0.00038	0.002	0.00035	0.00033	0.021	0.020
0.3	0.023	0.02064	0.01708	0.00061	0.00042	0.003	0.00037	0.00034	0.037	0.034
0.4	0.035	0.02272	0.01716	0.00091	0.00055	0.005	0.00040	0.00034	0.064	0.058
0.5	0.052	0.02548	0.01652	0.00157	0.00079	0.007	0.00043	0.00032	0.139	0.125
0.6	0.076	0.03629	0.01818	0.00628	0.00200	0.011	0.00059	0.00040	0.333	0.294
0.7	0.113	0.04568	0.01395	0.01216	0.00321	0.017	0.00077	0.00036	0.994	0.862
0.8	0.177	0.07757	0.01053	0.03814	0.01086	0.028	0.00136	0.00031	3.940	3.239
0.9	0.309	0.15720	0.01527	0.10309	0.04122	0.061	0.00521	0.00031	52.229	33.790

Table 2: Numerical comparison in terms of the average power

	$\delta = 1$					$\delta = 2$				
		Threshold		Power			Threshold		Power	
π_0	$\hat{\pi}_0$	Storey	Proposed	Storey	Proposed	$\hat{\pi}_0$	Storey	Proposed	Storey	Proposed
$\alpha = 0.05$										
0.1	0.385	0.01354	0.01390	0.11309	0.11506	0.140	0.31471	0.31475	0.93542	0.93544
0.2	0.454	0.00608	0.00639	0.06584	0.06813	0.237	0.14700	0.14709	0.82910	0.82919
0.3	0.522	0.00261	0.00288	0.03648	0.03914	0.332	0.08016	0.08025	0.72441	0.72462
0.4	0.591	0.00108	0.00130	0.01934	0.02215	0.428	0.04571	0.04583	0.62251	0.62295
0.5	0.658	0.00048	0.00062	0.01064	0.01292	0.522	0.02639	0.02651	0.52522	0.52601
0.6	0.728	0.00020	0.00030	0.00547	0.00748	0.619	0.01437	0.01447	0.42580	0.42690
0.7	0.796	0.00008	0.00015	0.00277	0.00436	0.714	0.00717	0.00728	0.32687	0.32880
0.8	0.864	0.00003	0.00007	0.00143	0.00248	0.809	0.00284	0.00294	0.22192	0.22539
0.9	0.933	0.00001	0.00003	0.00058	0.00130	0.905	0.00063	0.00072	0.11014	0.11763
$\alpha = 0.10$										
0.1	0.385	0.08648	0.08690	0.35840	0.35940	0.140	0.69094	0.69097	0.99376	0.99377
0.2	0.454	0.04558	0.04600	0.24531	0.24670	0.237	0.35102	0.35112	0.94711	0.94714
0.3	0.522	0.02289	0.02342	0.15928	0.16163	0.332	0.20450	0.20462	0.87987	0.87996
0.4	0.591	0.01071	0.01122	0.09671	0.09976	0.428	0.12363	0.12380	0.80038	0.80060
0.5	0.658	0.00462	0.00506	0.05448	0.05799	0.522	0.07569	0.07586	0.71407	0.71448
0.6	0.728	0.00185	0.00217	0.02854	0.03202	0.619	0.04380	0.04399	0.61479	0.61556
0.7	0.796	0.00067	0.00093	0.01363	0.01729	0.714	0.02341	0.02361	0.50478	0.50625
0.8	0.864	0.00022	0.00036	0.00605	0.00864	0.809	0.01010	0.01030	0.37343	0.37629
0.9	0.933	0.00006	0.00013	0.00214	0.00396	0.905	0.00260	0.00279	0.21342	0.22035